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Construction of the Outer Automorphism of $\mathcal{S}_6$ via a Complex Hadamard Matrix

Neil I. Gillespie · **Padraig Ó Catháin** ·
Cheryl E. Praeger

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Abstract We give a new construction of the outer automorphism of the symmetric group on six points. Our construction features a complex Hadamard matrix of order six containing third roots of unity and the algebra of split quaternions over the real numbers.

Keywords Complex hadamard matrix · Outer automorphism · Symmetric group

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1 Introduction

Sylvester showed that the fifteen two-subsets of a six element set can be formed into 5 parallel classes in six different ways and that the action of $\mathcal{S}_6$ on these *synthematic totals* is essentially different from its natural action on six points [13]. To our knowledge this was the first construction for the outer automorphism of $\mathcal{S}_6$.

Miller attributes the result that for $n \neq 6$, $\mathcal{S}_n$ has no outer automorphisms to Hölder, and Sylvester's construction of the outer automorphism of $\mathcal{S}_6$ to Burnside [11]. He also gives a by-hand construction of the outer automorphism. The papers of Janusz and Rotman, and of Ward provide easily readable accounts which are similar to Sylvester's [10, 14]. Cameron and van Lint devoted an entire chapter (their sixth!) to the outer automorphism of $\mathcal{S}_6$ [2]. They build on Sylvester's construction to construct the 5-(12, 6, 1) Witt design, the projective plane of order 4, and the Hoffman–Singleton graph.

Via consideration of the cube in $\mathbb{R}^3$, Fournelle gives a heuristic for the existence of an outer automorphism of $\mathcal{S}_6$, and constructs it with the aid of a computer [7]. Howard, Millson, Snowden and Vakil give two constructions of

N. I. Gillespie (✉)
Heilbronn Institute for Mathematical Research, University of Bristol, Tyndall Ave, Bristol BS8 1TH, UK
e-mail: neil.gillespie@bristol.ac.uk

P. Ó Catháin
Department of Mathematical Sciences, Worcester Polytechnic Institute, 100 Institute Road, Worcester, MA 01609, USA
e-mail: pocathain@wpi.edu

C. E. Praeger
School of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia
e-mail: cheryl.praeger@uwa.edu.au

the outer automorphism of $\mathcal{S}_6$, and use this to describe the invariant theory of six points in certain projective spaces [9].

In this note we give a construction which we believe has not previously been described, using a complex Hadamard matrix of order 6 and a representation of the triple cover of A_6 over the complex numbers. This note is inspired by a construction of Marshall Hall Jr [8] for the outer automorphism of M_{12} via a real Hadamard matrix of order 12, and by Moorhouse's classification of the complex Hadamard matrices with doubly transitive automorphism groups [12]. It was in the latter paper that we first became aware of the complex Hadamard matrix of order 6 discussed in this article, where it is described as corresponding to the distance transitive triple cover of the complete bipartite graph $K_{6,6}$.

2 Hadamard Matrices

Let ω be a primitive complex third root of unity. Then the matrix H_6 is *complex Hadamard*.

$$H_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \overline{\omega} & \overline{\omega} & \omega \\ 1 & \omega & 1 & \omega & \overline{\omega} & \overline{\omega} \\ 1 & \overline{\omega} & \omega & 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \overline{\omega} & \omega & 1 & \omega \\ 1 & \omega & \overline{\omega} & \overline{\omega} & \omega & 1 \end{pmatrix}$$

This means that H_6 satisfies the identity $H_6 H_6^{\dagger} = 6I_6$, where for an invertible complex matrix A, $A^{\dagger}$ is the complex conjugate transpose of A. Equivalently, H_6 reaches equality in Hadamard's determinant bound. We refer the reader to [6] for a comprehensive discussion of Hadamard matrices and their generalisations.

An *automorphism* of a complex Hadamard matrix is a pair of monomial matrices (P, Q) such that $P^{-1}HQ = H$. The set of all automorphisms of H forms a group under composition. In this note we will work with the subgroup of automorphisms (P, Q) where all non-zero entries are third roots of unity, we denote this group $\mathrm{Aut}(H)$. Consider now the projection maps $\rho_1(P, Q) \mapsto P$ and $\rho_2(P, Q) \mapsto Q$. Since $\frac{1}{\sqrt{6}}H_6$ is unitary, and for any automorphism (P, Q) of H the identity $HQH^{-1} = P$ holds, it follows that ρ_1 and ρ_2 are conjugate representations of $\mathrm{Aut}(H)$. Note further that ρ_i is a faithful representation, since $Q = I$ forces $P = I$. Thus $\mathrm{Aut}(H)$ is isomorphic to a finite subgroup of monomial matrices of $\mathrm{GL}_n(\mathbb{C})$. Furthermore, if $\mathrm{Aut}(H)$ contains a subgroup isomorphic to G, then the projections ρ_1 and ρ_2 onto the first and second components of $\mathrm{Aut}(H)$ give two conjugate representations of G by monomial matrices.

Every monomial matrix has a unique factorisation $P = DK$ where D is diagonal and K is a permutation matrix. The projection $\pi : P \mapsto K$ is a homomorphism for any group of monomial matrices. In general, the representation $\mathrm{Aut}(H)^{\rho_1\pi}$ is **not** linearly equivalent to the representation $\mathrm{Aut}(H)^{\rho_2\pi}$. As mentioned above, this phenomenon was first observed by Hall, who showed that the automorphism group of a Hadamard matrix of order 12 is isomorphic to $2 \cdot M_{12}$, and that $\rho_1\pi$ and $\rho_2\pi$ realise the two inequivalent actions of M_{12} on 12 points [8]. This interpretation of the outer automorphism of M_{12} was also used by Elkies, Conway and Martin in their analysis of the Mathieu groupoid M_{13} [4].

Throughout this note we use the following shorthand for monomial matrices: we list the elements of the diagonal matrix D, and give the cycle notation for K as a permutation of the **columns** of the identity matrix (i.e. a right action).

Consider the following pairs of monomial matrices.

$$\tau_1 := ([1, 1, 1, 1, 1, 1](2, 3, 4, 5, 6), \quad [1, 1, 1, 1, 1, 1](2, 3, 4, 5, 6))$$
$$\tau_2 := ([1, 1, \omega, \overline{\omega}, \overline{\omega}, \omega](1, 2), \qquad [1, 1, \overline{\omega}, \omega, \omega, \overline{\omega}]\,(1, 2)(3, 6)(4, 5))\,.$$

We define $*$ to be the entry-wise complex conjugation map, and consider the group $X = \langle \tau_1, \tau_2, * \rangle$.

Proposition 1 *The group X is of the form $3^{10} \cdot \mathcal{S}_6 \cdot 2$.*

Proof Since $\tau_1^* = \tau_1$ and $\tau_2^* = \tau_2^{-1}$, we have that $X_0 = \langle \tau_1, \tau_2 \rangle$ is normal in X. Hence $X = X_0 \rtimes \langle * \rangle$, with X_0 of index 2 in X.

The commutator $[\tau_2, *] = ([1, 1, \omega, \overline{\omega}, \overline{\omega}, \omega], [1, 1, \overline{\omega}, \omega, \omega, \overline{\omega}])$ consists of diagonal matrices; furthermore

$$\tau_2' := [\tau_2, *]^{-1}\tau_2 = ((1, 2), (1, 2)(3, 6)(4, 5)),$$

a pair of permutation matrices. Recall that $\langle s, t \mid s^6 = t^2 = (st)^5 = [t, s^2]^2 = [t, s^3]^2 = 1 \rangle$ is a presentation for $\mathcal{S}_6$ (see [1], for example). A computation with $t = \tau_2'$ and

$$s = \tau_1 \tau_2' = ((1, 2, 3, 4, 5, 6), (1, 2, 6)(3, 5))$$

shows that all the relations in this presentation hold for these elements s, t, and hence $Y = \langle \tau_1, \tau_2' \rangle$ is isomorphic to a quotient of $\mathcal{S}_6$. On the other hand, $Y^{\rho_1 \pi}$ is easily seen to be isomorphic to $\mathcal{S}_6$, so we conclude that $Y \cong \mathcal{S}_6$. Now let N be the subgroup of X consisting of all elements for which each component is a diagonal matrix. Since $\tau_1^{\rho_i}$ and $\tau_2^{\rho_i}$ have determinants in $\{\pm 1\}$, every element of the projection $X_0^{\rho_i}$ also has determinant ± 1. However all the elements of N^{ρ_i} have third roots of unity along the diagonal, and so must have determinant 1. As a result, $X_0^{\rho_i}$ is isomorphic to a subgroup of $M \rtimes \mathcal{S}_6$ where $M \cong 3^5$ is the group of unimodular diagonal matrices with entries from $\langle \omega \rangle$, and $\mathcal{S}_6$ acts as $Y^{\rho_i \pi}$. The only non-trivial $\mathcal{S}_6$-submodule of M is the constant module of order 3.

Define $n_{i+1} := [\tau_2, *]^{\tau_1^i}$ for each $i \geq 1$. (We shift subscripts because the action of τ_1 on $[\tau_2, *]$ gives elements of N which have the non-initial rows of H_6 as the diagonal of the first component.) Since $[\tau_2, *] \in X_0$, we have $n_i \in X_0$ for $2 \leq i \leq 6$. Observe that

$$\begin{aligned} n_3 n_4^2 n_5^2 &= ([1, 1, 1, 1, \omega, \overline{\omega}], \quad [1, 1, 1, 1, \overline{\omega}, \omega]) \\ (n_3 n_4^2 n_5^2)^{\tau_2'} &= ([1, 1, 1, 1, \omega, \overline{\omega}], \quad [1, 1, \omega, \overline{\omega}, 1, 1])\,. \end{aligned}$$

So neither of the projections N^{ρ_1}, N^{ρ_2} are onto the constant module, and the kernel of N^{ρ_1} is neither trivial nor the constant module. It follows that $N \cong M \times M$. Finally, we observe that monomial matrices normalise diagonal matrices, and that X_0 acts as a group of monomial matrices in each component. It follows that $N \lhd X_0$, and that Y is a complement of N in X_0. Since $*$ acts on N by inversion, $N \lhd X$. □

The group X has a natural action on 6×6 matrices over $\mathbb{C}$ where $(P, Q) \in X_0$ acts as $H^{(P,Q)} = P^{-1} H Q$, and $*$ acts by complex conjugation. We compute the stabiliser of H_6 under this action. We denote this group $\mathrm{Aut}^{\circ}(H_6)$ to emphasise that this is a group of semi-linear transformations in its action on the normal subgroup N. We require the subgroups X_0, Y and N defined in Proposition 1 in the proof of the following.

Proposition 2 *The group $\mathrm{Aut}^{\circ}(H_6)$ is isomorphic to the nonsplit extension $3 \cdot \mathcal{S}_6$, and $\mathrm{Aut}^{\circ}(H_6)$ contains a $\mathbb{C}$-linear subgroup isomorphic to $3 \cdot A_6$.*

Proof It is easily verified by hand that $H_6^{\tau_1} = H_6$ while $H_6^{\tau_2}$ is the complex conjugate H_6^*. Therefore both τ_1 and the product $\tau_2 *$ fix H_6. We claim that $\mathrm{Aut}^{\circ}(H_6) = \langle \tau_1, \tau_2 * \rangle$.

First, we show that the intersection $\mathrm{Aut}^{\circ}(H_6) \cap N$ has order 3. To prove this, suppose that $(D, E) \in N$, and that $D^{-1} H_6 E = H_6$, or equivalently $D H_6 = H_6 E$. Since the first column of H_6 is constant, D must be a scalar matrix. So D commutes with H_6, and we have $D H_6 = H_6 D = H_6 E$. Hence $D = E$, so $(D, E) = (\omega^i I, \omega^i I)$ for some i. Since these elements do leave H_6 invariant, the claim is proved.

We next claim that there is no element (D, E) of N such that $D H_6^* = H_6 E$; suppose to the contrary that such a (D, E) exists. Precisely the same argument as before shows that D must be scalar. This implies that $H_6^* = H_6 E D^{-1}$, but this equation has no solution in diagonal matrices: since the first row of H_6^* is equal to the first row of H_6, we would require $E D^{-1} = I_6$, from which we derive $H_6 = H_6^*$, a contradiction.

Consider the subgroup $K := \langle \tau_1, \tau_2 *, N \rangle$ of X. Since $X = \langle K, * \rangle$ and $* \notin K$, we have $|X : K| = 2$ and $X = K \cup (K *)$. It follows, moreover, from the previous arguments that no element of K sends H_6–H_6^*, and hence no element of the right coset $K*$ can fix H_6. Therefore, $\mathrm{Aut}^\circ(H_6) \subseteq K$, and from the first paragraph of the proof we also have $\mathrm{Aut}^\circ(H_6)N = K$. The quotient $\mathrm{Aut}^\circ(H_6)/(\mathrm{Aut}^\circ(H_6) \cap N)$ is isomorphic to K/N, an index 2 subgroup of $X/N \cong \mathcal{S}_6 \cdot 2$. In particular K/N contains A_6 as a normal subgroup of index 2. Since the element $N\tau_2*$ does not lie in A_6 and does not centralise A_6 it follows that $K/N \cong \mathcal{S}_6$.

We have shown that $\mathrm{Aut}^\circ(H_6)$ has a normal subgroup of order 3 with quotient isomorphic to $\mathcal{S}_6$. The elements $(\tau_2*)^{\tau_1^i}$ for $0 \leq i \leq 4$ project onto a set of Coxeter generators for $\mathcal{S}_6$. With these generators, it is straightforward to construct a Sylow 3-subgroup of $\mathrm{Aut}^\circ(H_6)$. One such subgroup is generated by

$$x := ([\overline{\omega}, 1, \omega, \omega, 1, \overline{\omega}](1, 2, 3), \quad [\omega, 1, \omega, 1, \overline{\omega}, \overline{\omega}](1, 4, 6)(2, 3, 5))$$
$$y := ([\omega, \overline{\omega}, 1, 1, \overline{\omega}, \omega](4, 5, 6), \quad [\omega, \omega, \omega, \omega, \omega, \omega](1, 4, 6)(2, 5, 3))\,.$$

A computation shows that $[x, y] = ([\omega, \omega, \omega, \omega, \omega, \omega], [\omega, \omega, \omega, \omega, \omega, \omega])$. This shows that the commutator subgroup contains the normal subgroup of order 3, hence the extension is non-split. Elements of $\mathrm{Aut}^\circ(H)$ which map onto odd permutations act on $[x, y]$ by inversion. So the centraliser of this normal subgroup is of index 2 in $\mathrm{Aut}^\circ(H)$: this is necessarily a non-split central extension $3 \cdot A_6$.

A perfect group S has a largest non-split central extension $\hat{S}$ which is unique up to isomorphism. The center of $\hat{S}$ is the Schur multiplier of S, and every non-split central extension of S is a quotient of $\hat{S}$. The number of generators of the Schur multiplier is bounded by $g - r$ where g is the number of generators in a presentation of S and r is the number of relations. We refer the reader to Wiegold's survey on the Schur multiplier for proofs of all these results [15]. Since A_6 is shown in [3] to have the presentation

$$\langle a, b \mid a^4, b^5, abab^{-1}abab^{-1}a^{-1}b^{-1}\rangle\,,$$

it follows that the Schur multiplier of A_6 is cyclic. Hence the non-split extension $3.A_6$ is unique up to isomorphism.

Now, since $\mathrm{Aut}^\circ(H)$ splits over $3 \cdot A_6$, we have that $3 \cdot A_6 < \mathrm{Aut}^\circ(H) < \mathrm{Aut}(3 \cdot A_6)$. Suppose that $\xi \in \mathrm{Aut}(3.A_6)$ such that the image of ξ in $\mathrm{Aut}(A_6)$ is the trivial automorphism. Let $\sigma \in 3.A_6$ be an element of order 15, projecting onto a 5-cycle in A_6. Then σ^5 generates the central subgroup of order 3. Each coset of $\langle\sigma^5\rangle$ contains a unique element of order 5, which is fixed by hypothesis. So either $\langle\sigma\rangle$ is fixed element-wise, or $\xi = *$. Moreover, any two subgroups of order 15 intersect in $\langle\sigma^5\rangle$, so the action of ξ is identical on all 5-cycles. Since the 5-cycles generate A_6, the action of ξ is completely determined.

So each choice of actions on 3 and on A_6 determines at most one isomorphism class of groups. It follows that $\mathrm{Aut}^\circ(H)$ is uniquely described as the group of shape $3.\mathcal{S}_6$ with trivial center.

The projection of $\rho_1(\mathrm{Aut}^\circ(H) \cap X_0)$ is clearly a faithful linear representation of $3.A_6$ over the complex numbers, completing the proof. □

In fact, $3.A_6$ is the largest subgroup of $\mathrm{Aut}^\circ(H_6)$ admitting a faithful 6-dimensional representation over $\mathbb{C}$. So this is $\mathrm{Aut}(H_6)$. A useful way to understand the actions of X and of $\mathrm{Aut}^\circ(H_6)$ is via a permutation action on 18 points, which we now describe. Let $P_1 = \tau_1^{\rho_1}$ and $P_2 = \tau_2^{\rho_1}$, and define the following 18×6 matrices:

$$M_1 = \begin{pmatrix} H \\ \omega H \\ \omega^2 H \end{pmatrix} \quad \text{and } M_2 = \begin{pmatrix} H^* \\ \omega H^* \\ \omega^2 H^* \end{pmatrix}.$$

For $1 \leq i \leq 18$, let $\mathrm{Row}_i(M_j)$ denote the ith row of M_j (where $j = 1, 2$). Let P_1 act on the rows of M_1, and similarly the rows of M_2, as follows:

$$P_1 \cdot M_1 = \begin{pmatrix} P_1 H \\ \omega P_1 H \\ \omega^2 P_1 H \end{pmatrix}$$

By letting P_2 act on the rows of M_1 and M_2 in a similar manner, we find that P_1 and P_2 act in the same way on the rows of M_1 and the rows of M_2, and hence act on the set $\Omega(18) := \{\{\mathrm{Row}_i(M_1), \mathrm{Row}_i(M_2)\} | i = 1, \ldots, 18\}$. Also, letting $*$ act as complex conjugation on M_1 and M_2, we see that $*$ also induces a permutation of $\Omega(18)$. Thus

τ_1, τ_2 and $*$ all induce permutations of $\Omega(18)$ and, identifying $\{\mathrm{Row}_i(M_1), \mathrm{Row}_i(M_2)\}$ with i, for each i, we get a permutation representation of X on 18 points with the following generating permutations:

$$\tau_1 = (2, 3, 4, 5, 6)(8, 9, 10, 11, 12)(14, 15, 16, 17, 18),$$

$$\tau_2 = (1, 2)(3, 15, 9)(4, 10, 16)(5, 11, 17)(6, 18, 12)(7, 8)(13, 14),$$

$$* = (7, 13)(8, 14)(9, 15)(10, 16)(11, 17)(12, 18).$$

The kernel of X in this action is the subgroup of N of order 3^5 consisting of pairs with trivial first component. The restriction to $\mathrm{Aut}^\circ(H_6)$ is faithful, however. One could construct a faithful action of X by taking the permutation action induced by its action on the rows of H_6 together with the induced action on columns.

Remark 3 The matrix H_6 and the group $3.A_6$ can be realised over any field k for which $k^\times$ has a subgroup of order 3. In the case that k is the finite field of order 4, the rows of H_6 span the *Hexacode*, introduced by Conway as part of a construction for the group M_{12}. It is discussed in detail in Sect. 11.2 of [5]. In particular, this code is the extended quadratic residue code with parameters (6, 3, 4). Uniqueness can easily be verified by hand: observe that the punctured code is the Hamming (5, 3, 3) code, which is unique, and that any pair of one-bit extensions which increase the minimum distance are isomorphic. The 6-dimensional $\mathbb{C}$-representation of $3 \cdot A_6$ has been previously described in the literature, normally via its action on a set of vectors in $\mathbb{C}^6$ derived from the hexacode. In particular, Wilson gives the action of $3 \cdot A_6$ on certain vectors of weight 4 in Sect. 2.7.4 of [16].

3 The Outer Automorphism of $\mathcal{S}_6$

Finally we construct the outer automorphism of $\mathcal{S}_6$ over the split-quaternions. Recall that the split-quaternions are a 4-dimensional $\mathbb{R}$-algebra with basis $[1, i, \beta, \beta i]$ where $[1, i]$ generates the usual algebra of complex numbers and $\beta^2 = 1$, $i^\beta = -i$. We denote the split quaternions by $\mathbb{B}$. They admit an $\mathbb{R}$-linear representation generated by

$$i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \beta \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Observe that $\mathrm{Aut}^\circ(H_6)$ admits a $\mathbb{B}$-linear representation if and only if $*$ does, and that the latter is realised by $(\beta I_6, \beta I_6)$.

Since H_6 is invertible over $\mathbb{C}$, it is invertible over $\mathbb{B}$. Now, rearranging the matrix equation $H_6^{\tau_2 *} = H_6$, and using the same notation as before for monomial matrices, we obtain that

$$H_6\,[[\beta, \beta, \beta\overline{\omega}, \beta\omega, \beta\omega, \beta\overline{\omega}]\,(1, 2)(3, 6)(4, 5)]\,H_6^{-1} = [[\beta, \beta, \beta\omega, \beta\overline{\omega}, \beta\overline{\omega}, \beta\omega]\,(1, 2)]\,.$$

Note that $(\beta\omega)^2 = (\beta\overline{\omega})^2 = 1$ so that the matrix on the right hand side of the above equation is an involution.

As was the case over the complex numbers, H_6 intertwines the projections ρ_1 and ρ_2. We observe that for any $g \in \mathrm{Aut}^\circ(H)$, we have that $g^{\rho_1} = H_6 g^{\rho_2} H_6^{-1}$. But, as illustrated above, $\tau_2^{\rho_1\pi}$ is a 2-cycle, while the projection $\tau_2^{\rho_2\pi}$ is a product of 3 disjoint 2-cycles. We conclude that the representations $\rho_1\pi$ and $\rho_2\pi$ of $\mathcal{S}_6$ cannot be conjugate. Thus whereas the permutation representations of $\mathcal{S}_6$ on 6 points are not equivalent, and the monomial representations of $3 \cdot A_6$ are not equivalent, we have constructed two explicit $\mathbb{B}$-linear representations of $3 \cdot \mathcal{S}_6$ which are equivalent under conjugation by H_6. Moreover, although the representation is not defined over $\mathbb{C}$, the intertwiner H_6 is.

Theorem 4 *There exists an irreducible* 6*-dimensional monomial representation of* $3 \cdot \mathcal{S}_6$ *over the split-quaternions. Two conjugate representations of* $3 \cdot \mathcal{S}_6$ *intertwined by the complex Hadamard matrix* H_6 *give an explicit construction for the outer automorphism of* $\mathcal{S}_6$.

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